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# DIFFRACTION OF ELASTIC WAVES BY THREE-DIMENSIONAL CRACKS OF ARBITRARY SHAPE IN A PLANE†

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A variational-difference method, proposed for solving two-dimensional integral equations of the convolution type in arbitrary regions [1], and highly recommended for solving dynamic contact problems [2, 3], is modified for the case of three-dimensional cracks. A general scheme of the method is given and ways of overcoming the difficulties that arise due to the singularity of the kernel, the increase in its symbol at infinity and taking into account the behaviour of the solution at the boundary of the region, are indicated. Calculations are carried out for rectangular and $L$-shaped cracks which show the effects of the shape of the crack, the angle of incidence, the type of incident wave and the frequency on the reflection coefficient, the radiation pattern and the redistribution of the energy in the reflected field. 

The analysis of the characteristics of the reflected wave field is a classical problem in geophysics, ultrasonic flaw detection, tomography, etc. In the short-wave band, when the dimensions of the obstacle considerably exceed the wavelength, the generalized ray method is employed for the successful mathematical modelling of the process [4], while in the case of medium and long waves it is necessary to solve boundary integral equations, which in the case of infinitely thin cracks considered here, reduces to Wiener–Hopf equations in the unknown jump in the displacements at the slit. In the plane case (a strip crack) and for circular cracks, theses equations are one-dimensional and they can be solved fairly effectively by expanding the unknown jump in displacements in orthogonal polynomials with a weight which takes into account the behaviour of the solution at the edges of the crack (detailed results for bulk waves incident on a circular crack at an arbitrary angle can be found, for example, in [5]).

For rectangular cracks it is necessary to use an expansion in Chebyshev polynomials in two spatial variables [6, 7], which involves increased computer costs due both to the increase in the dimension of the systems of relatively unknown coefficients of the expansion, and to the need to take into account double improper integrals when setting up the system matrix. At the same time, the rapid convergence, due to considering the root behaviour of the solution on the crack contour, is preserved. Unfortunately, this basis cannot be used for non-rectangular regions, and hence an approximation of the displacement jump by splines, specified in subregions in which the region occupied by the crack is divided into a certain mesh [8, 9], is used for a crack of arbitrary shape. The known behaviour at the edge was taken into account in [9] by introducing special boundary splines containing the root factor.

The method used in present paper can be regarded as a version of the general approach [8, 9], which enables the computing costs to be reduced considerably when the system is being set up by an appropriate choice of the form of the basis functions and by changing to single non-singular integrals. The approach described in [10], in which, as in [2], an axisymmetric basis is also proposed, is similar to ours. Unlike [10], we use another criterion of convergence and, for regions with rectangular perpendicular boundaries (rectangular, $L$-shaped, $\Pi$-shaped, etc. cracks) we propose an approximate method of taking the behaviour at the edge into account.

**1.** The problem of the diffraction of a specified wave field $\mathbf{u}_0(\mathbf{x})e^{-i\omega t}$ by a crack (an infinitely thin cut in elastic space with stress-free edges), occupying a region $\Omega$ in the $x, y$ plane of a Cartesian system of coordinates $\mathbf{x} = \{x, y, z\}$ can be reduced to solving a system of two-dimensional integral equations for the unknown jump in the displacements of its edges $\mathbf{v}(x, y) = \mathbf{u}(\mathbf{x})|_{z=0^+} - \mathbf{u}(\mathbf{x})|_{z=0^-}$

$$L\mathbf{v} \equiv \iint\limits_{\Omega} l(x-\xi, y-\eta)\mathbf{v}(\xi,\eta)d\xi d\eta = \mathbf{f}(x,y), \quad (x,y)\in\Omega \tag{1.1}$$

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where

$$l(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}L(\alpha_1,\alpha_2,\alpha)e^{-i(\alpha_1x+\alpha_2y)}d\alpha_1d\alpha_2,\quad \mathbf{f}(x,y)=-T_z\mathbf{u}_0|_{z=0}$$

and $T_z$ is the stress operator for the area with normal $z$.

In deriving (1.1) the well-known representation of the wave field in an elastic half-space in terms of its Green's matrix $k(x, y, z)$ and the vector of the unknown load $T_z\mathbf{u}|_{z=0}=\mathbf{q}(x,y)$ is used [3]

$$\mathbf{u}^{\pm}(\mathbf{x})=\frac{1}{4\pi^2}\int_{\Gamma_1}\int_{\Gamma_2}K^{\pm}(\alpha_1,\alpha_2,z)\mathbf{Q}(\alpha_1,\alpha_2)e^{-i(\alpha_1x+\alpha_2y)}d\alpha_1d\alpha_2 \tag{1.2}$$

Here and henceforth the superscript plus corresponds to $z > 0$, the superscript minus corresponds to $z < 0$, and $K^{\pm}$, $\mathbf{Q}$ are the Fourier symbols of $k^{\pm}$ and $\mathbf{q}$, respectively. In view of the discontinuity at $z = 0$ of the field of the reflected waves $u_1(\mathbf{x})$ it follows from (1.2) that

$$\mathbf{V}(\alpha_1,\alpha_2)=[K^+(\alpha_1,\alpha_2,0)-K^-(\alpha_1,\alpha_2,0)]\mathbf{Q}(\alpha_1,\alpha_2)$$

and, conversely

$$\mathbf{Q}(\alpha_1,\alpha_2)=L(\alpha_1,\alpha_2)\mathbf{V}(\alpha_1,\alpha_2) \tag{1.3}$$

where

$$L=[K^+-K^-]^{-1}|_{z=0},\quad \mathbf{V}(\alpha_1,\alpha_2)=\iint_{\Omega}\mathbf{v}(x,y)e^{i(\alpha_1x+\alpha_2y)}dxdy \tag{1.4}$$

For an homogeneous isotropic space

$$L(\alpha_1,\alpha_2,\alpha)=\begin{Vmatrix}(\alpha_1^2M_0+\alpha_2^2N_0)/\alpha^2 & \alpha_1\alpha_2(M_0-N_0)/\alpha^2 & 0\\ \alpha_1\alpha_2(M_0-N_0)/\alpha^2 & (\alpha_2^2M_0+\alpha_1^2N_0)/\alpha^2 & 0\\ 0 & 0 & R_0\end{Vmatrix}$$

$$M_0(\alpha)=-\Delta(\alpha)/(\varkappa_2^2\sigma_2),\quad N_0(\alpha)=-\mu\sigma_1/2,\quad R_0(\alpha)=-\Delta(\alpha)/(\varkappa_2^2\sigma_1)$$

$$\Delta(\alpha)=2\mu(\alpha^2\sigma_1\sigma_2-(\alpha^2-\varkappa_2^2)^2)$$

$$\sigma_n=\sqrt{\alpha^2-\varkappa_n^2},\quad \operatorname{Re}\sigma_n\geqslant 0,\quad \operatorname{Im}\sigma_n\leqslant 0,\quad n=1,2$$

$$\varkappa_1^2=\rho\omega^2/(\lambda+2\mu),\quad \varkappa_2^2=\rho\omega^2/\mu,\quad \alpha^2=\alpha_1^2+\alpha_2^2$$

where $\lambda$ and $\mu$ are Lamé constants and $\rho$ is the density.

The structure of the matrix $l$ here is such that system (1.1) can in fact be split into two independent matrices: in terms of the tangential and normal components of the jump $\mathbf{v}$. In the general case of a vertically-inhomogeneous space, for example, for a crack on the surface of a joint between two half-spaces with different properties, there is no such splitting, but the main properties of the elements of the matrix $L(\alpha_1, \alpha_2)$, which are essential for the proposed approach to be applicable, are preserved.

To discretize Eqs (1.1) axisymmetric delta-like splines, proposed earlier for solving dynamic contact problems [2, 3], are used. The approximate solution will be sought in the form

$$\mathbf{v}_h(x,y)=\sum_{k=1}^{N}\mathbf{c}_k\varphi_k(x,y),\quad \varphi_k(x,y)=\varphi\left(\frac{x-x_k}{h},\ \frac{y-y_k}{h}\right) \tag{1.5}$$

$$\varphi(x,y)=\begin{cases}\pi^{-1}(\gamma+1)(1-r^2)^{\gamma}, & r\leqslant 1\\ 0, & r\geqslant 1\end{cases},\quad r=\sqrt{x^2+y^2}$$

where $(x_k, y_k)$ are the nodes of a square grid, which covers the region $\Omega$ with a spacing $h$, and $N$ is the number of nodes.

The unknown expansion coefficients $c_k$ are found from the linear algebraic system which is obtained when the residual $L\mathbf{v}_n - \mathbf{f}$ is projected onto the same system of basis functions $\{\varphi\}_{j=1}^{N}$

$$\sum_{k=1}^{N} a_{jk}\mathbf{c}_k = \mathbf{f}_j, \quad j = 1,2,\ldots,N \tag{1.6}$$

$$a_{jk} = (L\varphi_k, \varphi_j)_{L_2}, \quad \mathbf{f}_j = (\mathbf{f}, \varphi_j)_{L_2}, \quad (f,g)_{L_2} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)g^*(x,y)dxdy$$

(the asterisk denotes complex-conjugate quantities).

It was shown in [3] that the fact that the functions $\varphi_k$ are delta-like, i.e. the fact that the condition

$$\frac{1}{h^2}\varphi\left(\frac{x}{h}, \frac{y}{h}\right) \to \delta(x,y) \quad \text{as} \quad h \to 0$$

is satisfied, ensures that the expansion coefficients $c_k$ will converge to the values at the nodes ($\mathbf{c}_k \to \mathbf{v}(x_k, y_k)$ as $h \to 0$), which enables us to ignore the convergence of $\mathbf{y}_h$ to $\mathbf{v}$ in a continuous metric (the best convergence of $c_k$ was obtained when $\gamma + 1 = \pi$, when $\varphi(0) = 1$). In other words, any deviation of $\mathbf{v}_h$ from $\mathbf{v}$ is permissible, and if it is necessary to obtain the form of the opening of the crack $\mathbf{v}(x, y)$, it is sufficient to use interpolation of the values $c_k$ between nodes. Here it is also easy to ensure the required behaviour at the boundary of $\Omega$. It is important that when using $\mathbf{v}_h$, in view of the fact that $\varphi_k$ is delta-like, that the convergence of the integral characteristics of the solution (the radiation pattern and the energy of the scattered field), should not be impaired.

In view of the fact that $\varphi = \varphi(r)$ is axisymmetric, the multiple integrals in (1.6) can be reduced, using Parseval's equality and changing to polar coordinates, to the single form

$$a_{jk} = \frac{h^4}{2\pi}\int_{\Gamma} L\left(\frac{i\partial}{\partial x_j}, \frac{i\partial}{\partial y_j}, \alpha\right)\Phi(\alpha h)\Phi^*(\alpha^* h)J_0(\alpha r_{jk})\alpha d\alpha \tag{1.7}$$

where

$$\Phi(\alpha h) = 2\pi\int_0^{\infty} \varphi(r)J_0(\alpha hr)rdr = \left(\frac{2}{\alpha h}\right)\alpha^{\gamma+1}\Gamma(\gamma+2)J_{\gamma+1}(\alpha h)$$

$J_\nu$ are Bessel functions and $r_{jk}$ is the distance between nodes.

Changing from the matrix-kernel $l(x, y)$ to its Fourier-symbol $L(\alpha_1, \alpha_2, \alpha)$ eliminates the need to separate and integrate the strong singularity of the matrix-kernel $l(x, y)$. The singular points (in the case considered these are $\varkappa_1$ and $\varkappa_2$) do not lie on the contour of integration since the contour $\Gamma$ bypasses them, deviating from the real semiaxis in the complex plane $\alpha$ in accordance with the principle of limiting absorption [3, 11]. However, difficulties connected with the poor convergence of the integrals (1.7) at infinity arise here. The elements $L$ increase as $\alpha \to \infty$ as $O(\alpha)$ (this also leads to a singularity of $l(x, y)$), and convergence is only obtained here due to the fact that $\Phi^2(\alpha h)$ decreases. When $h \ll 1$, the zone in which the decrease actually begins is shifted far to the right, which makes numerical integration practically impossible. Hence, the components of the functions $M_0$, $N_0$ and $R_0$, which occur in $L$, and which increases and decrease as $\alpha^{-1}$, were separated in explicit form, and the integrals of these were expressed in terms of hypergeometric functions ${}_3F_2$, i.e. they were represented by well-converging series.

An integral representation of the reflected wave field $u_1(x)$ in terms of the jump $\mathbf{v}$ found can be obtained by substituting (1.3) into (1.2). Here, by virtue of relation (1.5)

$$\mathbf{V}_h(\alpha_1, \alpha_2) = h^2\Phi(\alpha h)\sum_{k=1}^{N} \mathbf{c}_k e^{i(\alpha_1 x + \alpha_2 y)} \tag{1.8}$$

Taking into account the convergence $\mathbf{c}_k \to \mathbf{v}(x_k, y_k)$ as $h \to 0$ and the delta-form of the basis ($\Phi(\alpha h) \to \Phi(0)$ when $\alpha < \infty$, we can show that each of the terms corresponds in the limit to the contribution of an elementary area $h \times h$ with the centre at the node $(x_k, y_k)$, i.e. expansion (1.8) is the integral sum for $\mathbf{V}(\alpha_1, \alpha_2)$ of the form (1.4) for finite $\alpha$. As $\alpha \to \infty$ the decrease in $\mathbf{V}_h(\alpha_1, \alpha_2)$ does not correspond to the asymptotic form $\mathbf{V}(\alpha_1, \alpha_2)$, which can be written explicitly as the as asymptotic form of the oscillating integral (1.4), starting from the unknown form of the behaviour of $\upsilon$ on the boundary of $\partial\Omega$ [12], and taken into account in (1.8).

The behaviour of $\mathbf{v}$ on the boundary can be taken into account as follows:

1. by introducing root factors for the splines along the boundary;
2. by taking the asymptotic form of $\mathbf{V}$ as $\alpha \to \infty$ in (1.8);
3. by obtaining $c_k$ ignoring the behaviour on the boundary, then carrying out a smooth cubic integration between the nodes taking the root decrease at the edge into account and then using as $\mathbf{V}(\alpha_1, \alpha_2)$ a Fourier transformation of this approximation;
4. by arranging the choice of the nodes so that the effect of the root decrease of $v(x, y)$ is taken into account in the integral sum (1.8).

The first method destroys the axial symmetry, i.e. the corresponding computational advantages are lost, and the realization of the second approach is no simpler in practice. The third method considerably simplifies the results without destroying the axial symmetry. It is most important that the same degree of acceleration of the convergence should also be achieved by the fourth method, which is the simplest to realize.

The basic principle of this can be explained using the model one-dimensional integral

$$\int_0^b \sqrt{x}\,dx \approx h \sum_{k=1}^{M} \sqrt{x_k}, \quad x_1 = \frac{h}{2}, \quad x_{k+1} = x_k + h$$

The contribution of the first node $x_1$ of the quadrature formula equals $h^{3/2}\sqrt{2}$ whereas the exact value of the integral in the interval $[0, h]$ is equal to $2h^{3/2}/3$, i.e. the node $x_1$ introduces an error of $Oh^{3/2}$, whereas the inner nodes introduce an error of $Oh^2$. If we take $x_1 = h/2 + ph$, then for $p = 0.1133\ldots$ the contribution of $x_1$ is identical with the accurate value in the interval $[0, h + ph]$. The choice of a net with a space of precisely $ph$ from the boundary also ensures the same order of convergence as method 3. Note that this choice of the nodes is only possible for regions with perpendicular boundaries, when all the vertical and horizontal dimensions differ by an amount that is a multiple of $h$. If this is not so, the results will be obtained in fact for another region, which approximates the initial one and differs from it by dimensions not greater than $h/2$. Obviously, with this degree of convergence the method can also be used in the case of an arbitrary region $\Omega$, approximated by a rectangular grid.

Figure 1 shows a curve of the energy scattering coefficient $\Sigma$ as a function of the dimensionless frequency $\varkappa_2 a$ ($a = \sqrt{(S_\Omega)}/2$, and $S_\Omega$ is the area of the crack) for a quadratic crack for normal incidence ($\theta = 0°$) of a $P$-wave. The criterion for adjusting the method lies in the results obtained by Boström by expansion in Chebyshev polynomials as in [6] (the continuous curves). The small circles represent values of $\Sigma$ obtained by the fourth approach (a $20 \times 20$ grid). It should be noted that convergence, though

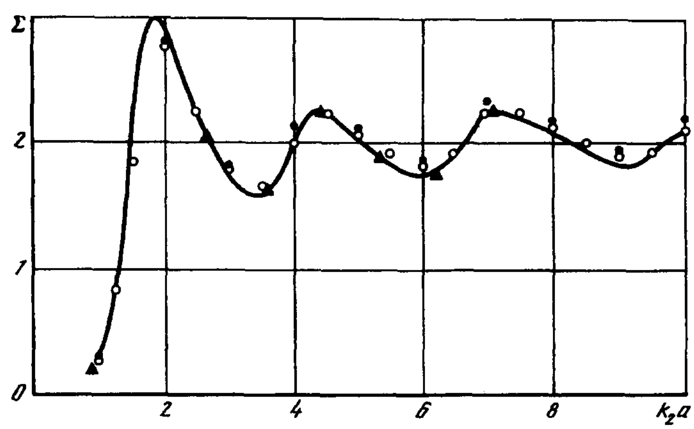


Fig. 1.

slower, is also observed ignoring the behaviour at the edge (the solid circles). The triangles in Fig. 1 represent the results obtained for a circular crack [5], which show that the shape of the crack in this case has only a small effect on the energy scattering coefficient.

**2.** The integral representation (1.2) enables the well-known asymptotic form of volume waves in an elastic half-space to be used in the far zone [3]

$$\mathbf{u}_1^\pm(\mathbf{x}) = \sum_{n=1}^{2} \mathbf{a}_n^\pm(\varphi,\psi)\frac{e^{i\varkappa_n R}}{R} + O(R^{-2}), \quad R = |\mathbf{x}| \to \infty \tag{2.1}$$

The vector functions $\mathbf{a}_n^\pm$, which depend only on the spherical angles $\varphi$ and $\psi$, are expressed in terms of the quantities $K^\pm LV$ at the stationary points $\alpha_{1,n} = -\alpha_n\cos\varphi$, $\alpha_{2,n} = -\alpha_n\sin\varphi$. Note that here we have used the values of $\mathbf{V}$ when $\alpha < \varkappa_n$, i.e. the error due to the fact that the asymptotic forms $\mathbf{V}$ and $\mathbf{V}_h$ are not identical has an effect only when $\varkappa_n \gg 1$, when ray methods work very well.

Since the Rayleigh denominator $\Delta(\alpha)$ is abbreviated in the product $K^\pm L$, then, as might have been expected, there is no Rayleigh wave along the surface $z = 0$ in $\mathbf{u}_1$. However, for a vertically-inhomogeneous half-space there are real poles in the elements of the matrix $L$, the residues in which give Stonely waves.

The energy of the reflected waves $E_1$, averaged over an oscillation period $T = 2\pi/\omega$, defined as the integral of the energy density over the surface of the edges of the crack which radiates it, reduces, using (1.2), to the well-known representation for the energy of a surface source in an elastic half-space, which in the final analysis gives

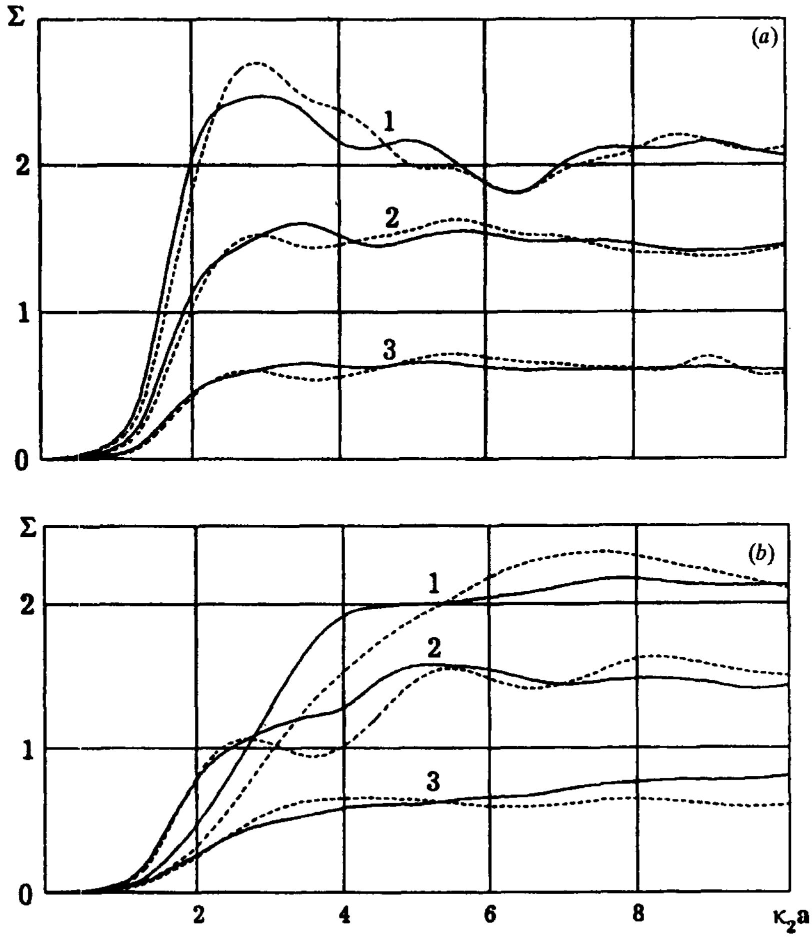


Fig. 2.

$$E_1 = -\frac{\omega}{2}\,\mathrm{Im}\iint_{\Omega}(\mathbf{f},\mathbf{v})\,dx\,dy = -\frac{\omega}{2}\,\mathrm{Im}\sum_{k=1}^{N}(\mathbf{f}_k,\mathbf{c}_k)$$

Similarly, the use of the asymptotic form (2.1) and the representations for the energy flux density of the longitudinal $P$-waves and transverse $S$-waves [3] enables us, by integrating them over the sphere $|\mathbf{x}| = R$ as $R \to \infty$, to obtain the fraction of the energy of the $P$ and $S$ waves in the flux $E_1$ scattered by the crack

$$E_1 = E_P + E_S \tag{2.2}$$

Satisfaction of the energy-balance equation (2.2) was used as an additional control of the numerical results. The calculations were carried out for rectangular and $L$-shaped cracks for $P$, $SV$ and $SH$ waves incident at angles from 0° to 90° in the frequency band $0 \leqslant a\varkappa_2 \leqslant 10$.

The degree of energy scattering is characterized by the ratio $\Sigma = E_1/E_0$, where $E_0$ is the energy transferred by the specified wave $u_0$ through an area equal to the area of the crack.

As an example, we show in Fig. 2 a graph of $\Sigma$ against the frequency $a\varkappa_2$ for a rectangular 1:4 crack (the dashed curve) and an $L$-shaped crack (the continuous curve) of the same area when a $P$ wave (a) and an $SV$ wave (b) are incident at angles of $\theta = 0°$, 45° and 90° to its normal (curves 1–3, respectively). We also obtained curves of $\Sigma$ against the angle of incidence $\theta$, and curves of the fraction of the energy of the $P$ and $S$ waves in the scattered field against $\theta$ and $\varkappa_2 a$. Figure 3, in which we show the results for a $P$-wave incident on a square crack at angles of $\theta$ and 0°, 45° and 90° (curves 1–3, respectively), illustrates how $E_S/E_1$ depends on the frequency. In Fig. 4, for an angle of incidence

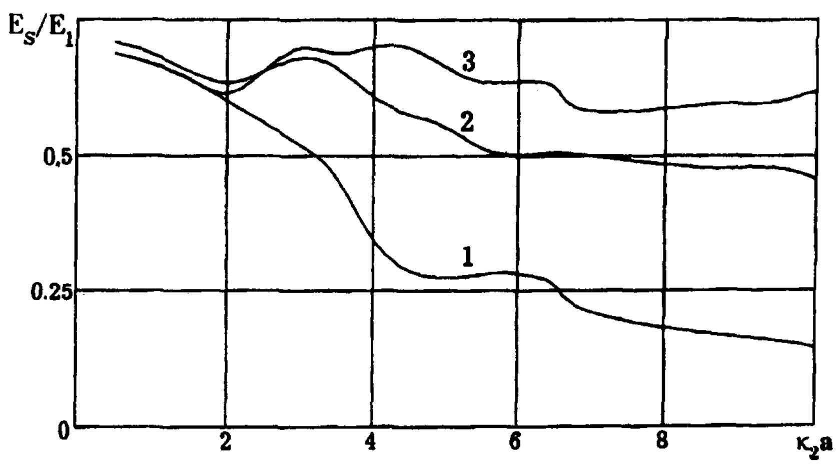


Fig. 3.

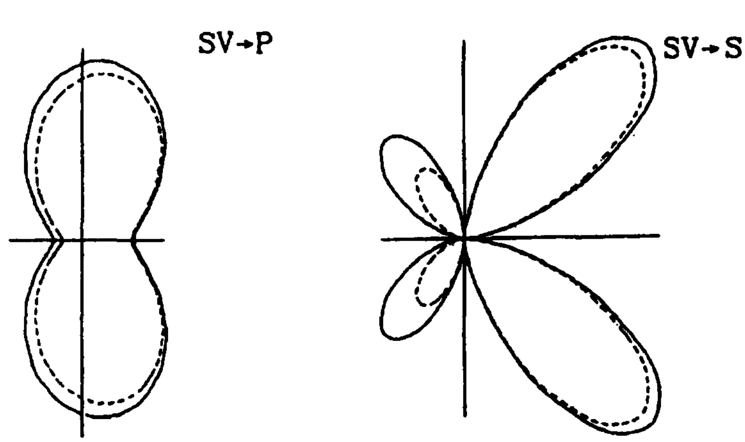


Fig. 4.

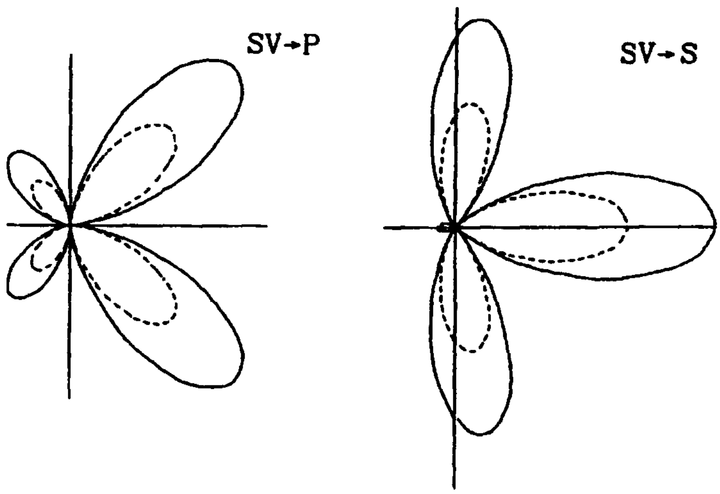


Fig. 5.

$\theta = 45°$, we show the radiation pattern of the scattered field (the energy densities of the $P$ and $S$ waves in the $xz$ plane) when an $SV$-wave is incident on a square crack (the continuous curves) and an $L$-shaped crack (the dashed curve) at a frequency $\varkappa_2 a = 2$ and in Fig. 5 for $\theta = 90°$.

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